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THIS STATISTICAL POINT ESTIMATION

FOR PARAMETERS OF SOME QUEUEING SYSTEMS

This paper deals with the problem of statistical point estimation for some queueing systems in equilibrium state with the first to come is the first to be served discipline.

The maximum likelihood method of estimation is used in this paper to estimate the rates of interarrival and service times and the utilization factor of each of those systems.

Finally there is a table including all the results obtained.

FORMULATION OF THE PROBLEM

Most probabilists generally divide the statistical problems into two types, "parameter estimation" and "distribution selection". In the case of the parameter estimation problems a particular type of probabilistic model is to prespecify an estimator for its parameter, while in the case of the distribution selection pro - lems appropriate data are to be examined as a basis for determining a choice of model. This second type of problems is the more important of the two, but us - ally also the more difficult. It is for that reason the overwhelming portion of literature has been concerned with parameter estimation .

The point estimation for some birth and death queueing models was co - sidered by wolf [7]. Later Clarke [3] dealt with the maximum likelihood estimation for some unknown parameters for the simple queue M/M/1. In the same year Benés considered this problem for the telephone exchange model.

This paper deals with the maximum likelihood point estimation for the rate of interarrival λ, the rate of service time u, and the utilization factor P of the m-server loss queue: M/M/m/m, the self regulating queue with finite customer population: M/M/1/M, the finite customer population queue with infinite number of servers: M/M/∞/M, the Erlangian queues: E_r/M/I and M/E_r/I, and at last the bulk arrival queue.

1. POINT ESTIMATION FOR THE M-SERVER LOSS QUEUE: M/M/m/m

The interarrival times and service times of this queueing system obey the exponential distribution or, equivalently, the arrival rate and service rate follow a Poisson distribution. Let A_k and U_k be the arrival and service rates respe - tively, then there are k customers in the system. There are m servers available in the system. Each newly arriving customer is given his private server; however if a customer arrives when all servers are occupied that customer is lost. Hence, $\lambda_\kappa=\lambda$, where λ is the mean interarrival time, and $U_k = K u$, if $K \leq m$, where U is the mean service time and $U_k = 0$ otherwise. Thus, the steady state probabilities $P_k \overset{\Delta}{=} \lim_{t \to \infty} P_k(t)$, where $P_k(t) \overset{\Delta}{=} P_r$ (k customers in the system at time t), are given as in [5] by

$$P_k = \begin{cases} (\frac{\lambda}{u})^k \frac{1}{k!} P_0 & \text{when } k \leq m \\ O & \text{otherwise,} \end{cases} \qquad 1.1$$

The expected number E (n) of customers in the system in the long rum is given by

$$E(n) = P_0 (\frac{\lambda}{u}) \left[\sum_{s=0}^{m-1} (\frac{\lambda}{u})^s \frac{1}{s!} \right] \qquad 1.2$$

This particular system is of great interest to those in telephony. Specially, P_m describes the fraction of time that all m servers are busy. The name given to this probability expression is Erlang's loss formula and it is given by

$$P_m = \frac{(\frac{\lambda}{u})^m / m!}{\left[\sum_{k=0}^{m} (\frac{\lambda}{u})^k \frac{I}{ki} \right]}$$

To find the likelihood function it is necessary to find the three basic components, namely:

(i) The initial number of units with contribution $Pr(\gamma)\, P\gamma$.

(ii) The interarrivals of length t_i (each of which is exponential for n arrival units with contribution $(\prod_{1=1}^{n} \lambda e^{-\lambda i})$.

(iii) The service time of duration t_i for k units with contribution $(\prod_{i=1}^{n} u e^{-ut_i})$.

Hence, the likelihood function may be written as

$$L(\Theta) = \prod_{1=1}^{n} \lambda e^{-\lambda t_1} \cdot \prod_{1=1}^{k} ue^{-ut_1} \cdot p_r(\lambda) : \quad \Theta \equiv (\lambda u)$$

$$= P_0 .e^{-\lambda t_1} .e^{-ur} .\lambda^{n+\gamma} .k_1$$

Where $T = \sum_{1=1}^{n} t_i$. $t = \sum_{1=1}^{n} t_i$. and K_1 is a constant independent of λ and u for all m.

Thus

ln [L (θ)] = ln Po - λT − τu + (n+ γ) ln λ + (κ − γ) ln u+ln k_1. [1.4]

By partial differentiation with respect to λ and u respectively and equating the results by zeros, it follows that

$$-\lambda \frac{\partial \ell nL}{\partial \lambda}\Bigg|_{\theta \equiv \hat{\theta}} = -\hat{\lambda} \left[\frac{\partial \ell np_0}{\partial \lambda} - T + \frac{n+\gamma}{\lambda}\right] = 0$$

and

$$u \frac{\partial \ell nL}{\partial \lambda}\Bigg|_{\theta = \hat{\theta}} = \hat{u}\left[\frac{\partial \ell np_0}{\partial u} - \tau + \frac{k-\gamma}{\hat{u}}\right] = 0$$

Where $\hat{\lambda}$ and û are the maximum likelihood estimators for the parameters λ and u respectively.

The following theorem is of special importance to this paper.

Theorem 1.1:

For the birth-death process with rates, as in wolf (7), namely $o \leq \lambda_n = \lambda g_n$, $o \leq u_n = u r_n$ for n positive integer and both g_n and r_n bounded functions of n. as in feller [4], we have

$$E(n) = -\lambda \frac{\partial \ell n p_0}{\partial \lambda} = u \frac{\partial \ell n p_0}{\partial u}.$$

Where $P_0 \triangleq$ delay probability and E (n) $\triangleq$ expected value of n.

Proof: As in Leonard [6], it is clear that

$$P_n = P_0 \prod_{l=0}^{n-1} \frac{\lambda_1}{u_{i+1}} \text{ for } n \geq I.$$

Where

$$P_0 = [1 + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \frac{\lambda_1}{U_{i+1}}]^{-1}$$

Case (1) : putting $\lambda_l = \lambda^n g_l$, it follows that

$$P_0 = [1 + \sum_{n=1}^{\infty} \lambda^n \prod_{i=0}^{n-1} \frac{g_i}{U_{i+1}}]^{-1}$$

Taking the logarithm of both sides of this equation and differentiating the resulting equation partially with respect to λ yield

$$-\frac{\partial \ell n p_0}{\partial \lambda} = p_0 [\sum_{n=1}^{\infty} n \lambda^{n-1} \prod_{i=0}^{n-1} \frac{g_i}{u_{i+1}}]$$

But,

$$E(n) = \sum_{n=1}^{\infty} n\, p_n = P_0 [\sum_{n=1}^{\infty} n \prod_{l=0}^{n-1} \frac{\lambda_i}{u_{i+1}}]$$

Therefore, it follows that

$$E(n) = -\lambda \frac{\partial \ell n p_0}{\partial \lambda}$$

Case (11) : putting $U_i = U_{ri}$ it follows that

$$P0 = [1 + \sum_{n=1}^{\infty} \frac{1}{u^n} \prod_{1=0}^{n-1} \frac{\lambda_i}{r_{i+1}}]^{-1}$$

Taking the logarithm on both sides of this equation, it follows that

$$\ell n P_0 = -\ell n [1 + \sum_{n=1}^{\infty} \frac{1}{u^n} \prod_{i=0}^{n-1} \frac{\lambda_i}{r_{i+1}}$$

Therefore,

$$\frac{\partial \ell n p_0}{\partial u} = P_0 . [\sum_{n-1}^{\infty} \frac{n}{u^{n+1}} \prod_{i=0}^{n-1} \frac{\lambda_i}{r_{i+1}}]$$

Hence,

$$u \frac{\partial \ell n p_0}{\partial u} = P_0 . [\sum_{n=1}^{\infty} n \prod_{i=0}^{n-1} \frac{\lambda_i}{r_{i+1}}]$$

Thus, it is evident that

$$E(n) = u \frac{\partial \ln p_0}{\partial u}$$

Hence, the theorem is proved.

Using this theorem yields

$$\left.\begin{aligned} \hat{\lambda} &= \frac{1}{T}[n + \gamma + \hat{E}(n)] \\ &\text{and} \\ \hat{u} &= \frac{1}{\tau}[k - \gamma + \hat{E}(n)], \end{aligned}\right\} \quad 1.5$$

Where Ê (n) is the maximum likelihood estimator of E (n).

Obtaining the value of Ê (n) from each of the equations in 1.5 and dividing the resulting values yields

$$\frac{\tau\hat{u} - (k-\gamma)}{-T\lambda + (n+\gamma)} = 1 \qquad [1.6]$$

Recalling that $\hat{p}$, the maximum likelihood estimator of the traffic intensity p, is given by $\hat{p} - \frac{\hat{\lambda}}{m\hat{u}}$ and using 1.5. it follows that

$$\hat{p} = \frac{\hat{\tau}(\eta + \gamma - \hat{E}(n))}{mT(k - \gamma + \hat{E}\ (n)} \qquad 1.7$$

substituting $\hat{E}$ (n) for its value into 1.7 yields

$$\tau[n + \gamma - m\hat{p} + \frac{(m\hat{p})^{m+1}}{m!}\hat{p}_0] - m\hat{p}T[k - \gamma + m\hat{p} - \frac{(m\hat{p})^{m+1}}{m!}\hat{p}_0] = 0 \qquad 1.8$$

Where $\hat{p}_0$ is the maximum likelihood estimator of P_0

The equation 1.8 is an equation in $\hat{p}$ of degree (m+1). Consider the special case M/M/1/1. On replacing $\hat{p}_0$ and m by $\frac{1}{1+\hat{p}}$ and 1 into 1.8, respectively, it follows that

$$\tau\,[(n-\gamma-\hat{p})\,(1+\hat{p}) + \hat{p}^2] - \hat{p}T\,[(k-\gamma+\hat{p})\,(1+\hat{p}) + \hat{p}^2] = 0$$

in other words

$$[T\,(k-\gamma+1)]\hat{p}^2 - [\tau(n+\gamma-1] - T(k-\gamma)]\hat{p} - \tau\,(n+\gamma) = 0 \qquad [1.9]$$

Replacing (K-γ-1) and (n+γ-1) by (k-γ) and (n + γ) respectively in 1.9 yields

$$(\hat{p} - 1)\,[T(k-\gamma)\ \hat{p} - \tau\,(n\ +\gamma)] = 0 \qquad [1.10]$$

Therefore,

$$P = \frac{\tau\,(n\ +\gamma)}{T\,(k-\gamma)} \qquad [1.11]$$

is a solution for the modified equation 1.10 of degree two in $\hat{p}$. It is clear that p has an F [2 (k- γ), 2 (n +γ)] distribution with known variance. Using 1.6 and 1.9, it follows that

$$\left.\begin{aligned} &\hat{u} = \frac{n+k}{T\hat{p}+\tau} = \frac{k-\gamma}{\tau} \\ \text{and} \\ &\lambda = \frac{\hat{p}(n+k)}{T\hat{p}+\tau} = \frac{n+\gamma}{T} \end{aligned}\right\} \quad [1.12]$$

2. POINT ESTIMATION FOR THE SELF REGULATING FINITE CUSTOMER POPULATION SINGLE SERVER QUEUE: M/M/1/M

The interarrival times and service times of this queueing system obey the exponential distribution. There is only a single queue and a single server. A customer is either in the system, or outside the system and in some sense "arriving". Let λ_k and u_k be the arrival and service rates, when there are k customers in the system. Customers act independently of each other. When there are k customers in the system, then there are (M-K) customers in the arriving state. Thus, the steady state probabilities

$p_k = \lim_{t \to \infty} p_{k(t)}$, where $p_k(t) \triangleq p_r$ (k customers in the system at the time t), are given as in (5) by:

$$P_k = \begin{cases} \left(\frac{\lambda}{u}\right)^k \frac{M!}{(M-K)!} P_0 & \text{For } 0 \le K \le M, \\ 0 & \text{Otherwise} \end{cases} \quad 2.1$$

Where

$$P_0 = \left[\sum_{k=0}^{M} \left(\frac{\lambda}{u}\right)^k \frac{M!}{(M-K)!}\right]^{-1}$$

It is evident that E(n), the expected number of customers in the system, is given by

$$E(n) = \sum_{n=0}^{m} n P_n$$

Therefore,

$$E(n) = P_0 \sum_{n=0}^{M} n \frac{M!}{(M-n)!} (\frac{\lambda}{u})^n$$

$$= P_0 \sum_{n=0}^{M} (n - M + M) \frac{M!}{(M-n)!} (\frac{\lambda}{u})^n$$

$$= P_0 \sum_{n=0}^{M} M \frac{M!}{(M-n)!} (\frac{\lambda}{u})^n - P_0 \sum_{n=0}^{M} \frac{M!}{(M-n-1)!} (\frac{\lambda}{u})^n$$

$$= P_0 (\frac{\lambda}{u})^{M+1} + M - \frac{\lambda}{u}$$

Thus, $\qquad E(n) = M - P + P^{M+1} P_0$

Where P, the utilization factor, is given by

$$p = \frac{\lambda}{u}$$

As in the previous section, the likelihood function is given by:

$$L(\Theta(= P_0 \;.\; \lambda^{n+\gamma} \;.\; u^{m-\gamma} \;.\; e^{-\tau u} \;.\; e^{-T\lambda} \;.\; K_2 \;;\; \Theta \equiv (\lambda, u)$$

Where K_2 is a constant independent of both λ and u. Therefore, it follows that

$$\left.\begin{aligned} -\lambda \frac{\partial \ell nL}{\partial \lambda}\Big|_{\Theta \equiv \hat{\Theta}} &= \hat{E}(n) + T\hat{\lambda} - (n+\gamma) \\ \text{and} \qquad & \\ u \frac{\partial \ell nL}{\partial u}\Big|_{\Theta \equiv \hat{\Theta}} &= \hat{E}(n) - \tau \hat{u} + (m-\gamma). \end{aligned}\right\} \qquad 2.3$$

Equating each of the equations in 2.3 by zero yields

$$\left.\begin{aligned} \hat{\lambda} &= \frac{1}{T}[n + \gamma - \hat{E}(n)] \\ \text{and} \qquad & \\ \hat{u} &= \frac{1}{\tau}[m - \gamma + \hat{E}(n)] \end{aligned}\right\} \qquad 2.4$$

Obtaining the value of $\hat{E}(n)$ from each of the equations in 2.4

Then dividing the results yield

$$\frac{T\hat{u} - m + \gamma}{n - T\hat{\lambda} + \gamma} = 1 \qquad 2.5$$

Recalling that $p = \frac{\lambda}{u}$ and using 2.4 , it follows that

$$\hat{p} = \frac{\tau[n + \gamma - \hat{E}(n)]}{T[m - \gamma + \hat{E}(n)]} \qquad 2.6$$

From 2.2 it is evident that

$$\hat{E}(n) = M - \hat{p} + \hat{p}^{M+1}\hat{p}_0$$

Where $\hat{E}(n)$, $\hat{p}$, and $\hat{p}_0$ are the maximum likelihood estimators of E(n), *P*, and P_0 respectively.

Replacing $\hat{E}(n)$ by its value in 2.6 yields.

$$[T(m - \gamma + M - \hat{p} + \hat{p}^{M+1}\hat{p}_0)]\hat{p} - [\tau(n + \gamma - M + \hat{\rho} M + 1\hat{P}_0)] = 0 \qquad 2.7$$

Equation 2.7 is an equation in $\hat{p}$ of degree (M+1). Consider the special case M/M/1/1. Letting M=1 in 2.7 and recalling that $p_0 = \frac{u}{u+\lambda}$ in M/M/1/1 queueing system , 2.7 reduces to the following equation:

$$[T(m - \gamma)]\hat{p}^2 + [T(m - \gamma + 1) - \tau(n + \gamma)]\hat{p} - \tau(n + \gamma - 1) = 0 \qquad 2.8$$

Replacing (n + γ −1) and (m - γ + 1) by (n + γ) and (m - γ) respectively in 2.8 yields.

$$(\hat{p} - 1)[\hat{p}T(m - \gamma) - \tau(n + \gamma)] = 0 \qquad 2.9$$

On solving the modified equation 2.9 , it follows that

$$p = \frac{\tau(n + \gamma)}{T(m - \lambda)} \qquad 2.10$$

is one of its solutions. *P* has an F [2 (m-γ), 2 (n + γ)] distribution with known variance using 2.5 and 2.10 , it follows that

$$\hat{u} = \frac{n+m}{T\hat{p}+\tau} = \frac{m-\gamma}{\tau}$$

and 2.11

$$\hat{\lambda} = \hat{p}\frac{n+m}{T\hat{p}+\tau} = \frac{n+\gamma}{\tau}$$

3- POINT ESTIMATION FOR THE FINITE CUSTOMER POPULATION AND INFINITE NUMBER OF SERVERS QUEUE: M / M / ∞ / M

The interarrival times and service times of this queueing system obey the exponential distribution. The population is considered to be finite. There is a single queue but an infinite number servers. Each newly arriving customer is given his private server. Let λ_k and u_k be the arrival rate and service rate, when there are k customers in the system.

Thus, the steady state probabilities $p_k \triangleq \lim_{t \to \infty} p_k(t)$. Where $p_k(t) \triangleq p_r$ (k customers in the system at time t), are given as in [5] by:

$$Pk = \begin{cases} (\frac{\lambda}{u})^k \frac{!}{k!} p_0 & o \le k \le M \\ O & \text{otherwise,} \end{cases} \quad 3.1$$

where

$$p_0 = (1+\frac{\lambda}{u})^{-M}$$

Denoting the expectation, that there are n customers in the system in the long run, by N, it follows that

$$N = \sum_{k=0}^{M} k\, p_k = \frac{1}{(1+\frac{\lambda}{u})^m} \sum_{k=0}^{M} (\frac{\lambda}{u})^k M \begin{bmatrix} M-1 \\ K-1 \end{bmatrix}$$

$$= \frac{(\lambda/u)}{(1+\frac{\lambda}{u})^M} \sum_{r=0}^{M-1} (\frac{\lambda}{u})^r \begin{bmatrix} M \\ r \end{bmatrix}$$

$$= \frac{(\lambda/u)}{(1+\frac{\lambda}{u})^{M}} \sum_{r=0}^{M-1} (\frac{\lambda}{u})^r \begin{bmatrix} M \\ r \end{bmatrix} - \frac{(\lambda/u)}{(1+\frac{\lambda}{u})M} (\frac{\lambda}{u})^M$$

Therefore,

$$N = \frac{\lambda}{u} - \frac{(\lambda/u)^{M+1}}{(1+\frac{\lambda}{u})_M} \qquad 3.2$$

Hence, it follows that

$$\hat{N} = \hat{\gamma} - \frac{\hat{\gamma} M + 1}{(1+\hat{\gamma})^M} \qquad 3.3$$

where $\hat{N}$ and $\hat{\gamma}$ are maximum likelihood estimators for the expectation N and the traffic intensity γ respectively.

As before, the likelihood function is given by

$L(\theta) = u^{m-\gamma} \; . \; \lambda^{n+\gamma} \; . \; e^{-u\tau} \; . \; e^{-\lambda T} \; . \; k_3;$ $\quad \Theta \equiv (\lambda, u)$

Where k_3 is a constant independent of λ and u. therefore, it follows that

$$\left. -\lambda \frac{\partial \ell nL}{\partial \lambda} \right|_{\theta=\hat{\theta}} = \hat{N} + T.\hat{\lambda} - (n+\gamma)$$

and $\qquad 3.4$

$$\left. u \frac{\partial \ell nL}{\partial \ell} \right|_{\theta=\hat{\theta}} = \hat{N} - \tau.\hat{u} + (m-\gamma)$$

Letting $\left. \frac{\partial \ell nL}{\partial \lambda} \right| \theta = \hat{\theta} = \left. \frac{\partial \ell nL}{\partial u} \right| \theta = \hat{\theta} = 0$ $\qquad$ 3.4 yields

$$\hat{\lambda} = \frac{n+\gamma-\hat{N}}{T}$$

and $\qquad 3.5$

$$\hat{u} = \frac{m-\gamma+\hat{N}}{\tau}$$

Thus, it is evident that

$$\frac{\tau\hat{u} - m + \gamma}{-T\hat{\lambda} + n + \gamma} = 1 \qquad 3.6$$

From 3.5 , it follows that

$$\hat{\gamma} = \frac{\tau(n+\gamma-\hat{N})}{T(m-\gamma+\hat{N})} = 1 \qquad 3.7$$

Using 3.3 and 3.7 , it is clear that

$$\hat{\lambda} = \frac{\tau\left[n+\gamma+\hat{\gamma}+\frac{\hat{\gamma}^{M+1}}{(1+\hat{\gamma})^{M}}\right]}{T\left[m-\gamma+\hat{\gamma}-\frac{\hat{\gamma}^{M+1}}{(1+\hat{\gamma})^{M}}\right]}$$

$$= \frac{\tau[(n+\gamma+\hat{\gamma})(1+\hat{\gamma})^{M}+\gamma^{M+1}]}{T[m-\gamma+\hat{\gamma})(1+\hat{\gamma})^{M}-\hat{\gamma}^{M+1}]}$$

Therefore,

$$[T(m-\gamma+\hat{\gamma})(1+\hat{\gamma})^{M}-\hat{\gamma}^{M+1}]\hat{\gamma}-[\tau\ (n+\gamma-\hat{\gamma})(1+\hat{\gamma})]=0 \qquad 3.8$$

Equation 3.8 is an equation in $\hat{\gamma}$ of degree (M+2)

Considering the special case M/M/∞/1. 3.8 reduces to

$$[T(m-\gamma+1)]\hat{\gamma}^{2}+[T(m-\gamma)-r(n+\gamma-1)]\hat{\gamma}-\tau(n+\gamma)=0 \qquad 1.9$$

Replacing (m - γ) and (n + γ) by (m - γ + 1) (n + γ – 1). 3.9 reduces to

$$(\hat{\gamma}+1)[\hat{\gamma}T(m-\gamma+1)-\tau(n+\gamma-1)]=0 \qquad 3.10$$

Solving this modified equation, it follows that

$$\gamma=\frac{\tau[n+\gamma-1]}{T[m-\gamma+1]} \qquad 3.11$$

is a solution of 3.10

It is evident that γ has an F[2(m-γ +1), 2(n + γ –1)] distribution with known variance. Using 3.6 and 3.11, it follows that

$$\hat{u}=\frac{n+m}{T\hat{p}+\tau}=\frac{m-\gamma+1}{\tau}$$

and 3.12

$$\hat{\lambda}=\hat{\gamma}\,\hat{u}=\frac{n+\gamma-1}{T}$$

4.POINT ESTIMATION FOR THE ERLANGIAN QUEUE : E_r/ M/1

The interarrival time obeys the Erlangian distribution whose probability density function, a (t), is given by

$$a(t) = \frac{r\lambda(r\lambda t)^{r-1} e^{-r\lambda t}}{(r-1)!} \qquad \text{for } t \geq 0.$$

But, the service time obeys the exponential distribution whose probability density function b(x) is given by

$$B(x) = u\, e^{-ux} \qquad \text{for } x \geq 0.$$

The arriving facility is an r stages Erlangian facility.

An arriving customer is immediately inserted from the left side of the arriving facility and passes through exponential stages each with parameter λ. On exiting from the right side of the arriving facility, an arrival to the given system E_r/M/1 is said to occur. No other customer can be inserted from the left side into the arriving facility before the previous one exits from the right side of this facility. Once having arrived, the customer joining the queue, waits for service and is then served according to the distribution b(x). the steady state probabilities $P_k = \lim_{t\to\infty} P_k(t)$, where $P_k(t) = P_r$ (K customers in the system at time t), are given as in (5) by

$$P_k = \sum_{J=rk}^{r(k+1)-1} p_j$$

Where $P_j = \lim_{t\to\infty} P_j(t)$ and $P_j(t) = P_r$ (j arrival stages in the system at time t). The probabilities P_j , j = 0,1,2, ... are derived from their z-transform P(z). The equilibrium equations can be written as

$$\left.\begin{array}{ll} r\lambda P_0 = u\, p_r & \\ r\lambda p_j = r\lambda p_{j-1} + u\, p_{j+r} & 1 \leq j \leq r-1 \\ (r\lambda + u)\, p_j = r\lambda p_{j-1} + u\, p_{j+r} & r \leq j. \end{array}\right\} \quad 4.1$$

operating upon the equations in 4.1 , adding and subtracting the missing terms as appropriate, yields

$$\sum_{j=1}^{\infty}(u+r\lambda)p_j z^j - \sum_{j=1}^{r-1} u p_j z^j = \sum_{j=1}^{\infty} r\lambda_{j-1} z^j + \sum_{j=1}^{\infty} u p_{j+r} z^j$$

Therefore

$$(u+r\lambda\,[p(Z)-p_0] - \sum_{j=1}^{r-1} u\,p_j\,z^j = r\,\lambda\,z\,P(z) + \frac{u}{2^r}[p(z) - \sum_{j=0}^{r} p_j\,z^j]$$

Thus, it is clear that

$$p(z) = \frac{(1-z^r)\sum_{j=0}^{r-1} p_j z^j}{r\,pz^{r+1} - (1+r\,p)^r z + 1}$$

It was explained by Leonard (5) that p(z) can be written as

$$p(z) = \frac{(1-z^r)}{k(1-z)(1-\frac{z}{z_0})}$$

Where k is a constant and Z_0 is a solution for the equation

$r\,p\,z^{r+1} - (1 + r\,p)\,z^r + 1 = 0$. But since p(1) = 1, it follows that

$$k = \frac{r}{1-\frac{1}{z_0}}$$

Therefore

$$p(z) = \frac{(1-z^r)\left[1-\frac{1}{z_0}\right]}{r\,(1-z)\,(1-\frac{z}{z_0})}$$

Operating upon the equations in 4.1 adding and subtracting the missing terms as appropriate, yields

$$\sum_{j=1}^{\infty}(u+r\lambda)\,p_j z^j - \sum_{j=1}^{r-1} up_j z^j = \sum_{j=1}^{\infty} r\lambda p_{j-1} z^j + \sum_{j=1}^{\infty} up_{j+r} z^j$$

Therefore

$$(u+r\lambda)[\,p(z)-p_0] - \sum_{j=1}^{r-1} up_j z^j = r\,\lambda\,z\,p\,(z) + \frac{u}{z^r}[p(z) - \sum_{j=0}^{r} p_j z^j]$$

Thus, it is clear that

$$p(z) = \frac{(1-z^r)\sum_{j=0}^{r-1} p_j z^j}{r\,pz^{r+1} - (1+rp)\,z^r + 1}$$

it was explained by Leonard (5) that p(z) can be written as

$$p(z) = \frac{(1-z^r)}{k\,(1-z)\,(1-\frac{z}{z_0})}$$

where k is a constant and Z_0 is a solution for the equation $r\,p\,z^{r+1} - (1 + r\,p)\,z^r + 1 = 0$, such that $|z_0| > 1$.

But since p (1) = 1, it follows that

$$k = \frac{r}{1-z_0}$$

therefore

$$p(z) = \frac{(1-z^r)\left(1-\frac{1}{z_0}\right)}{r\,(1-z)\,(1-\frac{z}{z_0})}$$

Thus,

$$p(z) = (1-z^r)\left[\frac{1/r}{1-z} - \frac{-1/rz_0}{1-\frac{z}{z_0}}\right] \qquad 4.2$$

Denoting the inverse z-transform of the quantity $\left(\frac{1/r}{1-z} + \frac{-1/rz_0}{1-\frac{z}{z_0}}\right)$ by f_j. It fo - lows that the inverse transform for p(z) must be $p_j = f_j - f_{j-r}$ 4.3

It is clear that

$$f_J = \begin{cases} \frac{1}{r}(1-z_0^{-J-1}) & J \geq 0 \\ 0 & \end{cases} \qquad 4.4$$

Recalling that Zo is a solution for the equation

$r\,p\,z^{r+1} - (1 + r.p)\,z^{r} + 1 = 0$

It follows that

$$r\,p(z_o - 1) = 1 - z_0^{-r} \qquad 4.5$$

using 4.3 , 4.4 , and 4.5 yield

$$p_j = \begin{cases} \frac{1}{r}(1 - z_0^{-J-1}) & \text{for } O \le J \le r \\ p(z_0 - 1)z_0^{r-J-1} & \text{for } J \ge r \end{cases} \qquad 4.6$$

Therefore, it is evident that

$$Pk = \begin{cases} (1-p) & \text{for } k = 0 \\ p(Z_0^{r} - 1)Z_0^{-rk} & \text{for } k > 0 \end{cases} \qquad 4.7$$

As before, the likelihood function is given by

$L(\Theta) = u^{m}.\lambda^{n}.e^{-u\tau}.p(Z_0^{r} - 1)Z_0^{-rk}\Theta \equiv (\lambda u)$

Therefore, it is evident that

$$\left.\frac{\partial \ell nL}{\partial u}\right|_{\Theta=\hat{\Theta}} = \frac{m-1}{\hat{u}} - \tau$$

and

$$\left.\frac{\partial \ell nL}{\partial \lambda}\right|_{\Theta=\hat{\Theta}} = \frac{n-1}{\hat{\lambda}} - t \qquad 4.8$$

where $\hat{\lambda}$ and $\hat{u}$ are the maximum likelihood estimation for λ and u. putting

$\left.\frac{\partial \ell nL}{\partial u}\right|_{\Theta=\hat{\Theta}} = 0 \text{ and } \left.\frac{\partial \ell nL}{\partial \lambda}\right|_{\Theta=\hat{\Theta}} = 0$ yields

$$\hat{\lambda} = \frac{n+1}{T} \quad and \quad \hat{u} = \frac{m-1}{\tau} \qquad 4.9$$

Therefore,

$$\hat{p} = \frac{\tau\,(n+1)}{T\,(m-1)} \qquad 4.10$$

$\hat{p}$, the maximum likelihood estimator for p, has an F[2 (m-1), 2 (n+1)] distrib - tion with known variance.

5. POINT ESTIMATION FOR THE ERLANGIAN QUEUE: M/E$_r$/1

The interarrival time obeys an exponential distribution with probability density function, a (t), is given by

$$a(t) = \lambda e^{-\lambda\tau} \qquad \text{for } t \geq 0$$

But, the service time obeys an Erlangian distribution with probability density function b(x), which is given by

$$b(x) = \frac{ru\,(r\,u\,x)^{r-1} e^{-rux}}{(r-1)!} \qquad x \geq 0$$

The service facility is an r stages facility. A customer enters the service facility, passes through exponential stages each with parameter ru, and finishes service on exiting from the right side of the service facility. No other customer can be inserted from the left side into the service facility before the previous one exits from the right hand side of this facility.

The steady state probabilities $P_k = \lim_{t \to \infty} p_k(t)$, where $p_k(t) = P_r$ (K customers in the system at time t), k = 1,2,... are given by

$$P_k = \sum_{J=(k-1)r+1}^{rk} p_j \qquad 5.1$$

where $P_j = \lim_{t \to \infty} P_j(t)$ and $P_j(t) = P_r$ (J service stages in the system at time t). The system state equations are given by

$$\left.\begin{aligned} \lambda p_0 &= r\,u\,p_1 \\ (r\,u + \lambda)\,P_j &= \lambda\,p_{j-r} + r\,u\,p_{j+1} \qquad j = 1,2,\ldots \end{aligned}\right\} 5.2$$

The method of the z-transform p(z) for the probabilities P_j is going to be used to find the steady state probabilities P_k ; k = 0.1,.... Multiplying the j th equation in 5.2 Z^j and summing over all applicable j yields

$$(\lambda + ru)\left(\sum_{j=0}^{\infty} p_j z^j - p_0\right) = \lambda z^r \sum_{j=1}^{\infty} p_{J-r} z^{J-r} + \frac{ru}{z}\sum_{j=1}^{\infty} z^{J+1} \qquad 5.3$$

Recalling that $p(z) = \sum_{J=0}^{\infty} p_J z^J$, equation 5.3 reduces to

$$p(z) = \frac{p_0 [\lambda + ru - \frac{ru}{z}] - rup_1}{(\lambda + ru - \lambda z^r - \frac{ru}{z})}$$

yielding finally

$$p(z) = \frac{rup_0 (1-z)}{ru + \lambda z^{r+1} - (\lambda + ru)^z}$$

where P_0 is a constant. Recalling that p(1) = 1, it can be easily deduced that p_0 = 1-p. substituting for P_0 by its value into p (z) yields.

$$p(z) = \frac{ru(1-p)(1-z)}{ru + \lambda z^{r+1} - (\lambda + ru)^z}$$

The denominator of P(z) has (r+1) zeros. 1 is one of its zeros. Thus the denom - nator of p(z) can be written as (1-z) [ru - λ $(z+z^2+\ldots+z^r)$], or it can be rewritten $ru(1-z)(1-\frac{z}{z_1})\ldots\ldots(1-\frac{z}{z_r})$, where 1, z_1, z_2, ..., z_r are the (r+1) roots of the denominator of p(z). making use of this result, p(z) can be rewritten as follows:

$$p(z) = \frac{(1-p)}{(1-\frac{z}{z_1})(1-\frac{z}{z_2})\ldots(1-\frac{z}{z_r})}$$

Hence,

$$P(z) = (1-P)\sum_{l=1}^{r} \frac{A_l}{(1-\frac{z}{z_l})}$$

where

$$A_l = \prod_{\substack{n=1 \\ n\neq l}}^{r} \frac{1}{(1-\frac{z_l}{z_n})} \qquad 5.4$$

since p_j is the coefficient of z^j in the expansion of p(z), then

$$P_j = \begin{cases} (1-p) & \text{for } j=0 \\ (1-p)\sum_{j=1}^{r} A_J (Z_1)^{-J} & \text{for } j = 1.2, \ldots\ldots r \end{cases} \qquad 5.5$$

As before, the likelihood function is given by

$$L(\Theta) = \lambda^n . (r-u)^{rm} . e^{-\lambda T} . \frac{\tau^{r-1}}{((r-1)!)^m} . e^{-rux} . (1-\frac{\lambda}{u}) . W_j; \Theta \equiv (\lambda, u)$$

where $W_j = \sum_{I=1}^{r} A_I (z_i)^{-J}$, $x = \sum_{I=1}^{m} x_I$, and both T and τ and are defined as previously done. Proceeding in the same way as in the previous section, it follows that

$$\frac{\partial \ell nL}{\partial \lambda}\Big|_{\Theta \equiv \hat{\Theta}} = \frac{n}{\hat{\lambda}} - T + \frac{1}{\hat{\lambda} - \hat{u}}$$

and

$$\frac{\partial \ell nL}{\partial \lambda}\Big|_{\Theta \equiv \hat{\Theta}} = \frac{rm-1}{\hat{u}} - rx + \frac{1}{\hat{u} - \hat{\lambda}}$$

where $\hat{\lambda}$ and $\hat{u}$ are the maximum likelihood estimators for λ and u respectively. Denoting $\frac{1}{\hat{\lambda} - \hat{u}}$ by $\hat{T}^{\bullet}$ and putting $\frac{\partial \ell nL}{\partial \lambda}\Big|_{\Theta = \hat{\Theta}} = 0$ and $\frac{\partial \ell nL}{\partial u}\Big|_{\Theta = \hat{\Theta}} = 0$ yield

$$\left.\begin{aligned} &\frac{n}{\hat{\lambda}} - T + \hat{T}^{\bullet} = 0 \\ &\text{and} \\ &\frac{rm}{\hat{u}} - rx - \hat{p}\hat{T}^{\bullet} = 0 . \end{aligned}\right\} \qquad 5.6$$

Obtaining the value of $\hat{T}^{\bullet}$ from each of the equations in 5.6 and dividing the resulting equations, it follows that

$$\frac{rm - rx\hat{u}}{T\hat{\lambda} - n} = 1 \qquad 5.7$$

from 5.6 , it follows that

$$\left.\begin{aligned} \hat{\lambda} &= \frac{\hat{p}+n(\hat{p}-1)}{T(\hat{p}-1)} \\ &\text{and} \\ \hat{u} &= \frac{r\,m(\hat{p}-1)-\hat{p}}{r\times(\hat{p}-1)} \end{aligned}\right\} \quad 5.8$$

Dividing the first equation by the second in 5.8 yields

$$[T(rm-1)]\,\hat{p}^{-2}-[T\,r\,m+x\,r(n+1)]\,\hat{p}+(r\times n)=0 \qquad 5.9$$

Replacing rm and n by (rm-1) and (n+1) respectively yields the following approximated solution:

$$p=\frac{x\,r\,(n+1)}{T\,(m-1)} \qquad 5.10$$

it is clear that p has an F[2(rm-1), 2r(n+1)] distribution.

From 5.7 and 5.10, it follows that

$$\hat{u}=\frac{n+r\,m}{T\hat{p}+r\,x}=\frac{r\,m-1}{r\,x}$$

and

$$\hat{\lambda}=\hat{p}\,\hat{u}=\frac{n+1}{T}$$

6. POINT ESTIMATION FOR THE BULK ARRIVAL QUEUE

In the $M/E_r/1$ queueing system, which has been studied in the previous section, each customer contributes r service stages on his arrival. The arrival of one customer can be considered as the arrival of r customers. The latter is an M/M/1 queueing system with "bulk" arrival. Consider that a random size bulk of probabilities g_l, l=0, 1, 2, ... is arrive at each arrival instant. As an example, one may think or random size families arriving at the doctor's office for individual vaccinations. Let λ_k and u_k be the arrival rate of bulks and their service rate when there are k customers in the system. The z-transform method is used to derive the probability distributions and the probability density functions.

Let E_k be the k^{th} state of the system. It is evident that the system can reach the state E_k from any lower state and can go out of the state E_k to any higher state, since the bulk size is random. The net rate at which the system leaves E_k is λ

The steady state probabilities $P_j = \lim_{t \to \infty} P_j(t)$, where $P_j(t)$ Pr (J customers in the system at time t), can be derived from their z-transform P(z). let G (Z) denote the z-transform of the bulk size, therefore

$$G(z) = \sum_{k=1}^{\infty} g_k z^k$$

The system equations are given by

$$\left.\begin{aligned} &\lambda P_0 = u P_1 \\ &(\lambda + u) p_k = u p_{k+1} + \sum_{l=1}^{k-1} p_j \lambda g_{k-l} \qquad \text{for } k \geq 1 \end{aligned}\right\} \quad 6.1$$

Multiplying the K^{th} equation in 6.1 by z^k and recalling that $p(z) = \sum_{k=0}^{\infty} p_k z^k$, it follows that

$$(\lambda + u) = \sum_{k=1}^{\infty} p_k z^k = \frac{u}{z} \sum_{k=1}^{\infty} P_{k+1} z^{k+1} + \sum_{k=1}^{\infty} \sum_{l=0}^{k-1} P_l \lambda g_{k-l} z^l \qquad 6.2$$

All the series in this equation are convergent, thus the two summation signs in the second term of the right hand side of 6.2 Can be exchanged. Therefore, it is evident that

$$\sum_{k=1}^{\infty} \sum_{l=0}^{k-1} p_l \lambda g_{k-l} z^l = \lambda p(z) G(z). \qquad 6.3$$

substituting by 6.3 into6.2 yields

$$p(z) = \frac{u p_0 (1-z)}{u(1-z) - \lambda z (1 - G(z))}$$

Recalling that P(1) = 1 it is evident that $P_0 = 1 - p$, where P is the utilization factor of our system; in other words P is the average arrival rate of customers multiplied by the average service time. In this section, the average arrival rate of customers is the product of the average arrival rate of bulks and the average bulk size. Thus, it is evident that

$$P = \frac{\lambda G'(1)}{u}$$

It can be easily shown that the distribution of the number of customers in the system is given by

$$P_J = (1-P)\sum_{l-1}^{k-1} A_l (Z_l)^{-J} \qquad 6.5$$

where $A_l = \prod_{\substack{n=1 \\ n\neq 1}}^{r} \frac{1}{(1-\frac{z_1}{z_2})}$ and $Z_1, Z_2, \ldots, Z_r$ are the roots of the following equation:

$$u(1-z) - \lambda z[1 - G(z)] = 0$$

As before, the likelihood function is given by

$$L(\theta) = \left(\prod_{l=1}^{m} [\lambda G'(1)] e^{-\lambda G'(1) t_l}\right) \times \left(\prod_{I=1}^{n} u\, e^{-u\tau_l} (1-p) A_l (Z_l)^{-J}\right)$$

Therefore.

$$L(\Theta) = [\lambda G'(1)]^m \quad e^{-\lambda G'(1)T} \quad u^n \quad e^{-u\tau} \quad |1 \quad \frac{\lambda G'(1)}{u}| \quad Wj$$

where $T = \sum_{I=1}^{m} t_l$, $\tau = \sum_{I=1}^{n} \tau_l$, and W_j is a constant. It is clear that

$$\frac{\partial \ell nL}{\partial \lambda} = \frac{m}{\lambda} - G'(1)T - \frac{G'(1)}{u(1-p)}$$

and

$$\frac{\partial \ell nL}{\partial u} = -\tau + \frac{n-1}{u} + \frac{1}{u - \lambda G'(1)}$$

Recalling that $p = \frac{\lambda}{u} G'(1)$, letting $\frac{\partial \ell nL}{\partial \lambda}|_{\Theta \equiv \hat{\Theta}} = 0$, and denoting $\frac{1}{\hat{u}(1-\hat{p})}$ by $\hat{T}^{\bullet\bullet}$ it follows that $\frac{m}{\hat{\lambda}} - G'(1)T - G'(1)\hat{T}^{\bullet\bullet} = 0$

And

$$-\tau+\frac{n-1}{\hat{u}}+\hat{T}^{**}=0. \qquad 6.6$$

where $\hat{\lambda}$, $\hat{u}$, and $\hat{p}$ are the maximum likelihood estimators for λ, u and *p* respectively. Using 6.6 yields

$$\hat{\lambda}=\frac{m(1-\hat{p})-\hat{p}}{G'(I)\,T\,(1-\hat{p})}$$

and 6.7

$$\hat{u}=\frac{n(1-\hat{p})+\hat{p}}{(1-\hat{p})}.$$

Finding the value of $\hat{T}^{**}$ from each equation in 6.6 and dividing the resulting equations by each other yield

$$\frac{\tau\,\hat{u}\,\hat{p}-(n-1)\hat{p}}{m-T\,\hat{u}\,\hat{p}}=1 \qquad 6.8$$

Recalling that $\hat{p}=\frac{G'(1)}{\hat{u}}$ and using 6.7 yield

$$(T\,(n-1))\,\hat{p}^2-(T\,n+\tau\,(m+1))\,\hat{p}+\tau m=0 \qquad 6.9$$

Replacing (n) and (m+1) by (n-1) and m respectively, it follows that

$$\hat{p}=\frac{\tau\,m}{T(n-1)} \qquad 6.10$$

is an approximated solution for 6.9.

form 6.8 and 6.10, it is evident that

$$\hat{\lambda}=\frac{\hat{u}\,\hat{p}}{G'(1)}=\frac{m}{T\,G'(1)}$$

and

$$\hat{u}=\frac{m+(n-1)\,\hat{p}}{\hat{p}\,(\tau+T)}=\frac{n-1}{\tau} \qquad 6.11$$

The following table includes all the results obtained.

The system	$\hat{\lambda}$	**û**	**Traffic intensity** $\hat{\gamma}$ **or utilization factor** $\hat{p}$
M/M/1/1 Single server loss system	$\frac{n+\gamma}{T}$	$\frac{k-\gamma}{\tau}$	$\hat{p}=\frac{\tau(n+\gamma)}{T(k-\gamma)}$
M/M/1/1 Self regulating system	$\frac{n+\gamma}{T}$	$\frac{m-\gamma}{\tau}$	$\hat{p}=\frac{\tau(n+\gamma)}{T(m-\gamma)}$
M/M/∞/1	$\frac{n+\gamma-1}{T}$	$\frac{m+\gamma-1}{\tau}$	$\hat{\gamma}=\frac{\tau(n+\gamma+1)}{T(m-\gamma+1)}$
Er/M/1	$\frac{n+1}{T}$	$\frac{m-1}{\tau}$	$\hat{p}=\frac{\tau(n+1)}{T(m-1)}$
M/Er/1	$\frac{n+1}{T}$	$\frac{rm-1}{rx}$	$\hat{p}=\frac{xr(n+1)}{T(m-1))}$
Bulk arrival system	$\frac{b}{TG'(1)}$	$\frac{n-1}{\tau}$	$\hat{p}=\frac{\tau m}{T(n-1))}$

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